On angular momentum of gravitational radiation

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Abstract

The quasigroup approach to the conservation laws (Phys. Rev. **D56**, R7498 (1997)) is completed by imposing new gauge conditions for asymptotic symmetries. Noether charge associated with an arbitrary element of the Poincaré quasialgebra is free from the supertranslational ambiquity and identically vanishes in a flat spacetime.

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I. INTRODUCTION

A major difficultes in defining of the angular momentim for isolated system are that, for asymptotically flat spacetimes the group of asymptotic symmetries is an infinite-parametric one (see,*e.g.*,[1] and references therein). The (asymptotic) symmetries group contains an unique four-parametric translation subgroup and infinite-parametric subgroup of supertranslations. Although the asymptotic symmetries group has a unique translation subgroup there is no canonical Lorentz subgroup since the last one emerges as a factor group of the asymptotic symmetries group by the infinite dimensional subgroup of supertranslations. This leads to the main problem in defining of angular momentum on null infinity, namely, all existing expressions suffer from supertranslational ambiguity.

In this letter we suggest a way to resolve this problem applying the quasigroup approach to the conservation laws in general relativity developed in [2, 3, 4, 5, 6, 7]. In Sec. 2 we outline the main facts from the theory of quasigroups of transformations and consider the asymptotic symmetries group at future null infinity ($\mathcal{J}^+$). In Sec. 3 new gauge conditions restricting the asymptotic symmeties group to a particulare Poincaré quasigroup are imposed and compared with the twistor approach [8, 9]. In Sec. 4 starting with the expression (linkage) given by Tamburino and Winicour [10, 11], we define the Noether charge associated with any element of the Poincaré quasialgebra and calculate the flux of the momentum and angular-momentum emitted by a gravitational system. Concluding remarks comprise the final Sec. 5.

We use the spin coefficients formalism by Newman and Penrose [8] choosing $\kappa = \varepsilon + \bar{\varepsilon} = 0,\ \tau = \bar{\pi} = \bar{\alpha} + \beta,\ \rho = \bar{\rho}$.

II. QUASIGROUPS OF TRANSFORMATIONS AND ASYMPTOTIC SYMMETRIES

Let $\mathfrak{M}$ be a n-dimensional manifold and the continuos law of transformation is given by $x' = T_a x, \quad x \in \mathfrak{M}$, where $\{a^i\}$ is the set of real parameters, $i = 1, 2, \ldots, r$. The set of transformations $\{T_a\}$ forms a r-parametric quasigroup of transformations (with right action on $\mathfrak{M}$), if [12]:

1) there exists a unit element which is common for all x^α and corresponds to $a^i = 0$:

$T_a x|_{a=0} = x$;

2) the modified composition law holds:

$$T_a T_b x = T_{\varphi(b,a;x)} x;$$

3) the left and right units coincide:

$$\varphi(a, 0; x) = a, \quad \varphi(0, b; x) = b;$$

4) the modified law of associativity is satisfied:

$$\varphi(\varphi(a, b; x), c; x) = \varphi(a, \varphi(b, c; T_a x); x);$$

4) the transformation inverse to T_a exists: $x = T_a^{-1} x'$.

The generators of infinitesimal transformations

$$\Gamma_i = (\partial(T_a x)^\alpha / \partial a^i)|_{a=0} \partial/\partial x^\alpha \equiv R_i^\alpha(x) \partial/\partial x^\alpha$$

form *quasialgebra* and obey the commutation relations

$$[\Gamma_i, \Gamma_j] = C_{ij}^p(x) \Gamma_p, \tag{1}$$

where $C_{ij}^p(x)$ are the structure functions satisfying the modified Jacobi identity

$$C_{ij,\alpha}^p R_k^\alpha + C_{jk,\alpha}^p R_i^\alpha + C_{ki,\alpha}^p R_j^\alpha + C_{ij}^l C_{kl}^p + C_{jk}^l C_{il}^p + C_{ki}^l C_{jl}^p = 0. \tag{2}$$

Theorem. Let the given functions R_i^α, C_{kj}^p obey the equations (1), (2), then locally the quasigroup of transformations is reconstructed as the solution of set of differential equations:

$$\frac{\partial \tilde{x}^\alpha}{\partial a^i} = R_j^\alpha(\tilde{x}) \lambda_i^j(a; x), \quad \tilde{x}^\alpha(0) = x^\alpha, \tag{3}$$

$$\frac{\partial \lambda_j^i}{\partial a^p} - \frac{\partial \lambda_p^i}{\partial a^j} + C_{mn}^i(\tilde{x}) \lambda_p^m \lambda_j^n = 0, \quad \lambda_j^i(0; x) = \delta_j^i. \tag{4}$$

Eq. (3) is an analog of the Lie equation, and Eq. (4) is the generalized Maurer-Cartan equation.

As it is known, for asymptotically flat at future null infinity ($\mathcal{J}^+$) spacetime the group of asymptotic symmetries is the infinite-parametric Newman-Unti (NU) group which contains the infinitedimensional Bondi-Metzner-Sachs (BMS) group preserving strong conformal geometry of $\mathcal{J}^+$ [13, 14, 15, 16, 17]. On $\mathcal{J}^+$ a general element of NU-algebra is given by

$$\xi = B(u, \zeta, \bar{\zeta}) \Delta^0 + C(u, \zeta, \bar{\zeta}) \bar{\delta}^0 + \bar{C}(u, \zeta, \bar{\zeta}) \delta^0, \tag{5}$$

where $\eth C = 0$ and $\eth, \Delta^0$, δ^0 are the standard NP operators "edth", Δ and δ restricted on $\mathcal{J}^+$; ζ being a complex stereographic coordinates at two-dimensional space-like cross sections of $\mathcal{J}^+$ labeled by a coordinate u and its metric is given by

$$ds^2 = \frac{2d\zeta d\bar{\zeta}}{[P(u,\zeta,\bar{\zeta})]^2}.$$

We assume $\zeta = x^2 - ix^3$ and that P can be written as

$$P = V(u,\zeta,\bar{\zeta})P_0,$$

where $P_0 = (1/\sqrt{2})(1+\zeta\bar{\zeta})$ and V is to be a regular function on the sphere. When $V = 1$ the metric above is reduced to the Bondi metric.

The generators of four-parameter translation subgroup read

$$\xi_a = B_a(\zeta,\bar{\zeta})\Delta^0, \quad (C = 0), \tag{6}$$

(a runs from 1 to 4) with the function B_a satisfying

$$\eth^2 B_a = B_a(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0), \quad \text{where} \quad \alpha^0 = \frac{1}{2}\bar{\eth}\ln P. \tag{7}$$

Its solution can be written as $B_a = l_a/V(u,\zeta,\bar{\zeta})$, where l_a satisfies $\eth_0^2 l_a = 0$ and a "standard edth" $\eth_0$ is referred to Bondi frame. There are four independent solutions of this equation.

The generators of "Lorentz group"are determined as follows:

$$\xi_A = B_A(u,\zeta,\bar{\zeta})\Delta^0 + C_A\bar{\delta}^0 + \bar{C}_A\delta^0, \quad (\eth C_A = 0), \tag{8}$$

(A runs from 1 to 6). Here $B_A(u,\zeta,\bar{\zeta})$ is an arbitrary real function and a dot being derivative with respect to retarded time u. There are six independent solutions of the equation $\eth C_A = 0$, which can be written as $C_A = l_A(\zeta,\bar{\zeta})/V(u,\zeta,\bar{\zeta})$, where l_A is a solution of the equation $\eth_0 l_A = 0$.

The generators of the NU group obey the commutation relations [7]:

$$\begin{aligned} [\xi_a, \xi_b] = 0, \quad [\xi_a, \xi_B] = C^b_{aB}(u,\zeta,\bar{\zeta})\xi_b, \\ [\xi_A, \xi_B] = C^D_{AB}(u,\zeta,\bar{\zeta})\xi_D, \end{aligned} \tag{9}$$

where C^b_{aB}, C^D_{AB} are the *structure functions* depending on an arbitrary function B_A. The commutation relations above show that the NU group is in fact a *quasigroup* with the closed Lorentz quasialgebra and a supertranslational ambiguity.

III. REDUCTION OF THE NU GROUP TO THE POINCARÉ QUASIGROUP

The key idea is to reduce the NU group to the ten-parametric quasigroup (the Poincaré quasigroup), imposing the appropriate conditions on an arbitrary function B_A, and thus fixing the supertranslational freedom.

The scheme consists of following [4, 6, 7]:

(i) Propagate the asymptotic generators ξ inward along the null superface Γ intersecting $\mathcal{J}^+$ in Σ^+ by means of geodesic deviation equation:

$$\nabla_l^2 \xi + R(\xi, l)l = 0, \tag{10}$$

imposing the appropriate conditions at $\mathcal{J}^+$.

(ii) Use the commutation relations (9) and asymptotic expansion of C^a_{bc}

$$C = C_0 + C_1 r^{-1} + C_2 r^{-2} + \cdots ,$$

r being a canonical parameter and C_0-s the same as in Eq.(9), evaluate all coefficients of these series.

In the spin-coefficient notation the geodesic deviation equation

$$\nabla_l^2 \xi + R(\xi, l)l = 0, \tag{11}$$

accomplished by substituting

$$\xi = AD + B\Delta + \bar{C}\delta + C\delta, \tag{12}$$

reads

$$\begin{aligned} D^2 A + 2(\tau D\bar{C} + \bar{\tau} DC) + CD\bar{\tau} + \bar{C} D\tau - \\ -C\bar{\Psi}_1 - \bar{C}\Psi_1 - B(\Psi_2 + \bar{\Psi}_2) = 0, \end{aligned} \tag{13}$$

$$D^2 B = 0, \quad D^2 C + BD\tau + \bar{C}\Psi_0 + B\Psi_1 = 0., \tag{14}$$

Using the asymptotic expansion

$$A = A_1 r + A_0 + A_{-1} r^{-1} + 0(r^{-2}),$$

$$B = B_1 r + B_0 + B_{-1} r^{-1} + 0(r^{-2}),$$

$$C = C_1 r + C_0 + C_{-1} r^{-1} + 0(r^{-2}),$$

one obtains the solution of the geodesic deviation equation in the following form:

$$A_{-1} = \Re(B\Psi_2^0 - 2\tau^0 \bar{\eth} B), \quad C_{-1} = B\tau^0, \quad B_{-n} = 0 \ (n \geq 1). \tag{15}$$

The system of equations (13), (14) is the second order ordinary differential system, and, for obtaining the unique solution, one needs to impose initial conditions on the functions A, B, C and its first derivatives. So the only freedom in the solution is in $A_0, A_1, B_0, B_1, C_0, C_1$. We adapt the asymptotic Killing equations for determining these coefficients. Starting with

$$\lim_{r\to\infty} l^\mu l^\nu \pounds_\xi g_{\mu\nu} = 0, \quad \lim_{r\to\infty} m^\mu n^\nu \pounds_\xi g_{\mu\nu} = 0, \quad \lim_{r\to\infty} m^\mu \bar{m}^\nu \pounds_\xi g_{\mu\nu} = 0, \tag{16}$$

$$\lim_{r\to\infty} r m^\mu \bar{m}^\nu \pounds_\xi g_{\mu\nu} = 0, \quad \lim_{r\to\infty} r l^\mu m^\nu \pounds_\xi g_{\mu\nu} = 0, \tag{17}$$

one obtains

$$A_0 = \eth\bar{\eth} B + B\eth\bar{\eth} \ln P, \quad A_1 = -\frac{1}{2}(\bar{\eth} C_1 + \eth \bar{C}_1), \quad B_1 = 0 \Rightarrow B = B_0, \tag{18}$$

$$C_0 = -\eth B + \sigma_0 \bar{C}_1, \quad \dot{C}_1 + (\ln P)\dot{}\, C_1 = 0 \Rightarrow C_1 = c(\zeta\bar{\zeta})/V, \tag{19}$$

(dot being derivative with respect to retarded time u) and the remaining freedom is in the functions B, C_1.

Now let us consider in the flat spacetime all null hypersurfaces and not just those which are shear-free. Then the functions B, C_1 must satisfy the following equations:

$$\eth^2 B - B(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0) - \frac{\sigma^0}{2}(3\eth\bar{G}_1 - \bar{\eth} C_1) - \bar{C}_1 \eth\sigma^0 - C_1 \bar{\eth}\sigma^0 = 0, \tag{20}$$

$$\eth C_1 = 0, \tag{21}$$

(σ^0 being the asymptotic shear), arising from the Killing equation

$$m^\mu m^\nu \pounds_\xi g_{\mu\nu} = 0. \tag{22}$$

The system above is the unique one determining the functions B, C_1 and restricting the NU group to a particular Poincaré group. Its solution is given by

$$B = B_t + \left(\eth\eta\bar{C}_1 + \frac{u-\eta}{2}\eth\bar{C}_1 + c.c.\right) \tag{23}$$

where

$$\sigma^0 = \eth^2\eta - \eta(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0), \quad \text{with} \quad \eta = u - \frac{1}{V}\int_0^u V du, \tag{24}$$

and B_t satisfies Eq.(7):

$$\eth^2 B_t = B_t(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0), \tag{25}$$

Setting $B = \xi^0 + \bar{\xi}^0, C_1 = \bar{\xi}^1$, where

$$\xi^0 = \xi^0_t + \eth\eta\xi^1 + \frac{u-\eta}{2}\eth\xi^1,$$

we find that ξ^0, ξ^1 satisfy

$$\eth^2\xi^0 = \xi^0(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0) + \frac{1}{2}\sigma^0\eth\xi^1 + \eth(\sigma^0\xi^1), \quad \bar{\eth}\xi^1 = 0.$$

In the presence of radiation the equation (22) leads to $\eth C_A = 0$ and

$$\eth^2 B - B(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0) - B\bar{\mathcal{N}} = \frac{\sigma^0}{2}(3\eth\bar{G}_A - \bar{\eth}C_A) + \bar{C}_A\eth\sigma^0 + C_A\bar{\eth}\sigma^0,$$

where the news function $\mathcal{N} = \lambda^0 + 2\bar{\eth}\alpha^0 - 4(\alpha^0)^2$ is introduced. In general case the solution of this equation does not exist, because B is a real function and σ^0 is the complex function. Thus the equation above is incompatible with (23), unless $\mathcal{N} = 0$.

A way to resolve this problem is to obtain the differential constraints on the functions B and C_1, which are compatible with Eq.(22) for the flat(stationary) spacetimes and lead to the definition of the asymptotic shear as $\sigma_0 = \eth^2\eta - \eta(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0)$ for some complex function η, so next we devote our attention to a method that fixes this problem.

Let us consider at $\mathcal{J}^+$ the complex vector

$$\xi_c = \xi^0\Delta^0 + \xi^1\delta^0, \tag{26}$$

such that the generators (5) read $\xi = \xi_c + \bar{\xi}_c$. This yields $B = \xi^0 + \bar{\xi}^0,\ \bar{C}_1 = \xi^1$ and $\bar{\eth}\xi^1 = 0$. We specify the vector ξ_c as follows:

$$\xi^0 = \xi^0_t + \eth\eta\xi^1 + \frac{u-\eta}{2}\eth\xi^1, \quad \eth^2\eta - \eta(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0) = \sigma^0, \tag{27}$$

where ξ^0_t is the solution of the following equation:

$$\eth^2\xi^0_t = \xi^0_t(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0), \tag{28}$$

determining the translational subgroup of the NU group. The definition (27) implies that ξ^0 satisfies

$$\eth^2\xi^0 = \xi^0(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0) + \frac{1}{2}\sigma^0\eth\xi^1 + \eth(\sigma^0\xi^1), \tag{29}$$

This differential equation is compatible with the Killing equations (see discussion above) and its solution (27) exists in general case, including radiation. We adopt (29) as the master equation restricting the NU group to a particular Poincaré quasigroup.

Emerging of the constraint (29) also can be understood analyzing the twistor equation on a cross section S of $\mathcal{J}$

$$\nabla_{A'}{}^{(A}\omega^{B)} = 0, \tag{30}$$

projected on S [18, 19] . The components of the equation (30) tangent to S are

$$\bar{\eth}\omega^0 = 0, \quad \eth\omega^1 = \sigma^0\omega^0. \tag{31}$$

Let ω_1^A, ω_2^A be any two solutions of (31). We associate with the twistor $\omega^{AB} = \omega_1^{(A}\omega_2^{B)}$ the complex NU generator

$$\xi_c = \xi^0\Delta^0 + \xi^1\delta^0 = -i(\omega_1^0\omega_2^1 + \omega_2^0\omega_1^1)\Delta^0 - 2i\omega_1^0\omega_2^0\delta^0. \tag{32}$$

With help of (31) one obtains that $\bar{\eth}\xi^1 = 0$ and ξ^0 satisfies

$$\eth^2\xi^0 = \xi^0(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0) + \frac{1}{2}\sigma^0\eth\xi^1 + \eth(\sigma^0\xi^1), \tag{33}$$

which is just (29). In the Bondi reference frame (33) reduces to the equations obtained by Shaw (Eq.(20) in [19]).

Our results can be summarized as follows:

(i) A general element of NU-algebra is given by

$$\xi = B(u,\zeta,\bar{\zeta})\Delta^0 + C(u,\zeta,\bar{\zeta})\bar{\delta}^0 + \bar{C}(u,\zeta,\bar{\zeta})\delta^0, \quad \eth C = 0, \tag{34}$$

$$B = B_t + \eth\eta\bar{C}_1 + \frac{u-\eta}{2}\eth\bar{C}_1 + \bar{\eth}\bar{\eta}C_1 + \frac{u-\bar{\eta}}{2}\bar{\eth}C_1,$$

$$\eth^2\eta - \eta(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0) = \sigma^0, \quad \eth^2 B_t = B_t(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0). \tag{35}$$

(a) The generators of translations are given by

$$\xi_a = B_a\Delta^0, \tag{36}$$

where $B_a = l_a(\zeta,\bar{\zeta})/V(u,\zeta,\bar{\zeta})$, and l_a satisfies $\eth_0^2 l_a = 0$.

(b) The generators of boosts and rotations are given by

$$\xi_A = B_A\Delta^0 + C_A\bar{\delta}^0 + \bar{C}_A\delta^0, \quad \eth C_A = 0, \tag{37}$$

$$B_A = \eth\eta\bar{C}_A + \frac{u-\eta}{2}\eth\bar{C}_A + c.c., \quad \eth^2\eta - \eta(4(\bar{\alpha}^0)^2 - 2\eth\bar{\alpha}^0) = \sigma^0, \tag{38}$$

where $C_A = l_A(\zeta,\bar{\zeta})/V(u,\zeta,\bar{\zeta})$, and l_A satisfies $\eth_0 l_A = 0$.

(ii) The generators of the Poincaré quasigroup obey at $\mathscr{J}^+$ the following commutation relations:

$$[\xi_a,\xi_b] = 0, \quad [\xi_a,\xi_B] = C^b_{aB}(u,\zeta,\bar{\zeta})\xi_b,$$
$$[\xi_A,\xi_B] = C^D_{AB}(u,\zeta,\bar{\zeta})\xi_D,$$

$C^D_{AB},\ C^i_{aB}$ being the *structure functions.*

IV. ENERGY-MOMENTUM AND ANGULAR MOMENTUM AT $\mathscr{J}^+$

As it is known the Komar integral [20], providing a fully satisfactory notation of the total mass in stationary, asymptotically flat spacetimes, is not invariant under a change of the choice of the generators of time translations in the equivalence class associated with the given BMS translation. For an asymptotically flat at future null infinity spacetime the modified "gauge invariant" Komar integral (linkage)

$$L_\xi(\Sigma) = -\lim_{\Sigma_\alpha\to\mathscr{J}^+}\frac{1}{4\pi}\oint_{\Sigma_\alpha}\left(\xi^{[\alpha;\beta]} + \xi^\rho_{;\rho} l^{[\alpha}n^{\beta]}\right)ds_{\alpha\beta}, \tag{39}$$

where $\{\Sigma_\alpha\}$ is one-parameter family of spheres, was introduced by Tamburino and Winicour [10]. We adopt this as our definition of the conserved quantities on $\mathscr{J}^+$ associated with the generators of the Poincaré quasigroup. The computation leads to the following coordinate independent expression

$$L_\xi = -(1/4\pi)\Re\oint\left\{B\left(\Psi^0_2 + \sigma^0\lambda^0 - \eth^2\bar{\sigma}^0\right)\right.$$
$$\left. +2\bar{C}\left(\Psi^0_1 + \sigma^0\eth\bar{\sigma}^0 + (1/2)\eth(\sigma^0\bar{\sigma}^0)\right)\right\}dS,\ (\eth C = 0). \tag{40}$$

Introducing the complex Noether charge

$$Q_c(\xi) = -(1/4\pi)\oint\left\{\xi^0\left(\Psi^0_2 + \sigma^0\lambda^0 - \eth^2\bar{\sigma}^0\right)\right.$$
$$\left. +\xi^1\left(\Psi^0_1 + \sigma^0\eth\bar{\sigma}^0 + (1/2)\eth(\sigma^0\bar{\sigma}^0)\right)\right\}dS \tag{41}$$

and setting $B = \xi^0 + \bar{\xi}^0,\ C_1 = \bar{\xi}^1$, we find

$$L_\xi = Q_c(\xi) + \bar{Q}_c(\xi). \tag{42}$$

Now applying *Lemma 2* and *Lemma 3* [18]:

$$\oint \xi^0 \left(\eth^2\bar{\sigma}^0 + \left(2\eth\bar{\alpha}^0 - 4(\bar{\alpha}^0)^2\right)\bar{\sigma}^0\right) dS + \oint \xi^1 \left(\sigma^0\eth\bar{\sigma}^0 + \frac{1}{2}\eth(\sigma^0\bar{\sigma}^0)\right) dS = 0,$$

$$\oint \xi^0 \left(\bar{\eth}^2\sigma^0 + \left(2\eth\bar{\alpha}^0 - 4(\alpha^0)^2\right)\sigma^0\right) dS = 0$$

together with the Bianchi identity $\Psi_2^0 + \sigma^0\lambda^0 - \bar{\eth}^2\sigma^0 = \bar{\Psi}_2^0 + \bar{\sigma}^0\bar{\lambda}^0 - \eth^2\bar{\sigma}^0$, one can write

$$Q_c(\xi) = -(1/4\pi)\oint \left(\xi^0\left(2\Psi_2^0 + 2\sigma^0\mathcal{N} - \bar{\Psi}_2^0 - \bar{\sigma}^0\bar{\mathcal{N}}\right) + \xi^1\Psi_1^0\right) dS, \tag{43}$$

This immediatly yields $Q_c \equiv 0$ in flat spacetime.

The integral four-momentum is given by [4, 6, 7]

$$P_i = -(1/4\pi)\Re\oint B_i(\Psi_2^0 + \sigma^0\mathcal{N})dS, \tag{44}$$

where $B_i = l_i/V$ and four-vector

$$l = \frac{1}{1+|\zeta|^2}\left(1+|\zeta|^2, \zeta+\bar{\zeta}, i(\zeta-\bar{\zeta}), |\zeta|^2-1\right).$$

Using the Bianchi identities, we compute the loss of energy-momentum

$$\dot{P}_i = -(1/4\pi)\oint B_i|\mathcal{N}|^2 dS. \tag{45}$$

This expression coincides with the result obtained by Lind *et al* [21] and in the Bondi frame it yields the standard formula (see, *e. g.* [8])

$$\dot{P}_i = -(1/4\pi)\oint l_i|\lambda^0|^2 dS. \tag{46}$$

The angular momentum is given by

$$M_A = -(1/4\pi)\Re\oint \bar{l}_A\mathcal{K}dS, \tag{47}$$

where l_A is the solution of $\eth l_A = 0$, and

$$\begin{aligned}\mathcal{K} = 2\Psi_1^0 + 3\eth\eta(2\Psi_2^0 + 2\sigma^0\mathcal{N} - \bar{\Psi}_2^0 - \bar{\sigma}^0\bar{\mathcal{N}})\\ +(\eta - u)\left(2\eth\Psi_2^0 + 2\eth(\sigma^0\mathcal{N}) - \eth\bar{\Psi}_2^0 - \eth(\bar{\sigma}^0\bar{\mathcal{N}})\right).\end{aligned} \tag{48}$$

The loss of angular momentum is found to be

$$\dot{M}_A = -(1/4\pi)\Re\oint \bar{l}_A\mathcal{G}dS, \tag{49}$$

where we set $\mathcal{G} = V^3\partial/\partial u(\mathcal{K}/V^3)$.

Let Σ_1 and Σ_2 be arbitrary cross-sections at $\mathcal{J}^+$, then the flux of the energy-momentum and angular momentum is given by

$$P_i(\Sigma_2) - P_i(\Sigma_1) = -(1/4\pi)\oint B_i|\mathcal{N}|^2 dSdu, \tag{50}$$

$$M_A(\Sigma_2) - M_A(\Sigma_1) = -(1/4\pi)\Re\oint \bar{l}_A\mathcal{G}dSdu, \tag{51}$$

where integration is performed over the domain $\Omega \subset \mathcal{J}^+$ contained between Σ_1 and Σ_2.

V. CONCLUDING REMARKS

Note that the group of isometries can be defined also as a group which transforms an arbitrary geodesic to a geodesic one and the Killing vectors satisfy the geodesic deviation equation for any geodesic. In our construction above (Sec. 3) only *null* geodesics passing inward are transformed to the geodesics under the transformations of the Poincaré quasigroup. For non-radiating at $\mathcal{J}^+$ systems the generators the Poincaré quasigroup is isomorphic to the Poincaré group. It agrees with the well known results on the reduction of the BMS group to the Poincaré group for the asymptotically flat stationary spacetime.

Our definition of the Noether charge associated with an arbitrary elements of the Poincaré quasialgebra is free from the supertranslational ambiguity and yields identically vanishing charge/flux in a flat spacetime. It essentially depends on the geometry of future null infinity and the behavior of generators near of $\mathcal{J}^+$. The integral conserved quantities $P(\xi,\Sigma)$ and a flux integral, giving the difference $P(\xi,\Sigma') - P(\xi,\Sigma)$ for the cross-sections Σ' and Σ, are linear on generators of Poincaré quasigroup and defined for system with radiation on $\mathcal{J}^+$.

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